Anti-pluricanonical systems on Fano varieties

Preliminaries II

2.31. Non-klt centres.

Lemma 2.33. *Assume that*

- *(X, B) is an lc pair,*
- *$G \subset X$ is a subvariety with normalisation F,*
- *X is $\mathbb{Q}$-factorial near the generic point of G, and*
- *there is a unique non-klt place of (X, B) whose centre is G.*

(ν_1, ν_2 two non-klt places w/ centre G $\Rightarrow \nu_1 \sim \nu_2$)

(X, B can have other non-klt places with centre not G)

Then if (Y, B_Y) is a $\mathbb{Q}$-factorial dlt model of (X, B) and S is a component of $\lfloor B_Y \rfloor$ mapping onto G, then the induced morphism $h\colon S \to F$ is a contraction. Moreover, the only non-klt centre of (X, B) containing G is G itself.

$$\begin{array}{ccc} Y \supset B_Y & & S \text{ is a comp } \lfloor B_Y \rfloor \\ \downarrow & & \downarrow \\ X \supset B & & G \end{array}$$

<u>Proof</u> Assume $\dim G < \dim S$. In particular, S is exc over X.
Let Π be the fibre of $Y \to X$ over a gen point of G.
Connectedness principle $\Rightarrow \lfloor B_Y \rfloor$ is connected near Π.

Let R be the support of $(\lfloor B_Y \rfloor - S)|_S$.
The morphism $R \to F$ is not surj.

T is contained in S. In particular, T is the fibre of h over g, hence h has connected fibres.
This implies h is a contraction.

2.34. **Pseudo-effective thresholds.**

Lemma 2.35. *Let $\mathcal{P}$ be a log bounded set of log smooth projective pairs (X, B). Then there is a number $\lambda > 0$ such that if $(X, B) \in \mathcal{P}$ and if K_X is not pseudo-effective, then $K_X + \lambda B$ is not pseudo-effective.*

Proof Replacing $\mathcal{P}$, can assume there is a sm proj morphism $f: V \to T$ of smooth var's and a reduced div S on V w/ snc singularities s.t. if $(X, B) \in \mathcal{P}$, then X is a fibre of f over some closed pt and supp B is inside $S|_X$. Can assume $B = S|_X$. We can assume S and $K_V + S$ are ample over T.

Let

$$t = \inf\{s \mid K_V + sS \text{ is ps-eff over } T\}.$$

Can assume $t > 0$. Run MMP on K_V. Fix $0 < \lambda < t$. F be gen fibre of f. Then $K_F + \lambda S|_F$ is not pseff.

2.36. Numerical Kodaira dimension. D $\mathbb{R}$-Cartier on normal proj var X. Replace X by a resolution and replace D by its pullback. If D not ps-eff, $\kappa_\sigma(D) = -\infty$. Otherwise, $\kappa_\sigma(D)$ is the largest int r s.t.

$$\limsup_{m\to\infty} \frac{h^0(\lfloor mD\rfloor + A)}{m^r} > 0$$ for some ample Cartier div A.

If D eff $\Rightarrow$ $\lfloor mD\rfloor + A$ is big $\Rightarrow$ $\kappa_\sigma(D) \geq 0$.

Lemma 2.37. *Let $\mathcal{P}$ be a bounded set of smooth projective varieties X with $\kappa_\sigma(K_X) = 0$. Then there is a number $l \in \mathbb{N}$ such that $h^0(lK_X) \neq 0$ for every $X \in \mathcal{P}$.*

Proof Can replace $\mathcal{P}$ s.t there is a sm proj morph $f: V \to T$.
By Deformation inv of log plurigenera, $\kappa_\sigma(K_F) = 0$ for every fibre F of f. Apply thm of Gongyo and semi-continuity of cohomology to show that $\exists\, l \in \mathbb{N}$: $h^0(lK_F) \neq 0$ for every fibre F.

2.38. Volume of divisors. D $\mathbb{R}$-divisor, X normal proj var, $\dim X = d$.

$$\mathrm{Vol}(D) := \limsup_{m\to\infty} \frac{h^0(\lfloor mD \rfloor)}{m^d/d!}.$$

Lemma 2.39. *Let X be a $\mathbb{Q}$-factorial normal projective variety of dimension d, D be an $\mathbb{R}$-divisor with $\kappa_\sigma(D) > 0$, and A be an ample $\mathbb{Q}$-divisor. Then* $\lim_{m\to\infty} \mathrm{vol}(mD + A) = \infty$.

Proof Assume X is smooth. D ps-eff $\Rightarrow$ $D = P_\sigma(D) + N_\sigma(D)$, $P_\sigma(D)$ is ps-eff, $N_\sigma(D) \geq 0$ [Nakayama].

Can assume $N_\sigma(D) = 0$. Let C be a curve cut out by gen members of $|lA|$ for $l \gg 0$. Can assume C does not intersect $\mathrm{Bs}\,|\lfloor mD \rfloor + A|$ of any m. In particular, $\exists$ resolution $\phi: W \to X$ s.t. movable part M of $|\phi^*(\lfloor mD \rfloor + A)|$ is base point free and support of fixed part F does not intersect $\phi^{-1} C$.

Then

$$\mathrm{vol}(mD + 2A) \geq \mathrm{vol}(\lfloor mD \rfloor + 2A) = \mathrm{vol}(\phi^*(\lfloor mD \rfloor + 2A))$$

$$\geq \mathrm{vol}(M + \phi^* A) \geq M \cdot (\phi^* A)^{d-1} = \phi^*(\lfloor mD \rfloor + A) \cdot (\phi^* A)^{d-1}$$

$$= (\lfloor mD \rfloor + A) \cdot C.$$

Since $\kappa_\sigma(D) > 0$, we have $D \cdot C > 0$, hence $(\lfloor mD \rfloor \cdot A) \cdot C$ is not bounded. Hence $\mathrm{vol}(mD + 2A)$ is not bounded.

Lemma 2.40. *Let $d \in \mathbb{N}$ and let $\mathcal{P}$ be a set of pairs (X, A) where X is smooth projective of dimension $\leq d$ with $\kappa_\sigma(K_X) > 0$ and A is very ample. Then for each $q \in \mathbb{N}$ there is $p \in \mathbb{N}$ such that for every $(X, A) \in \mathcal{P}$ we have* $\operatorname{vol}(pK_X + A) > q$.

Proof Use previous Lemma.

2.41. **The restriction exact sequence.**

$$0 \to \mathcal{O}_X(-S) \to \mathcal{O}_X \to \mathcal{O}_S \to 0$$

$$0 \to \mathcal{O}_X(L-S) \to \mathcal{O}_X(L) \to \underbrace{\mathcal{O}_X(L) \otimes_{\mathcal{O}_X} \mathcal{O}_S}_{=: \mathcal{F}} \to 0$$

Lemma 2.42. *Assume (X, B) is dlt for some boundary B and that S is $\mathbb{Q}$-Cartier. Let U be the largest open subset of X on which L is Cartier. If the codimension of the complement of $S \cap U$ in S is at least two, then $\mathcal{F} \simeq \mathcal{O}_S(L|_S)$.*

2.43. Descent of nef divisors.

X Z

Lemma 2.44. *Let $f\colon X \to Z$ be a contraction from a smooth projective variety to a normal projective variety with rationally connected general fibres. Assume M is a nef Cartier divisor on X such that $M \sim_{\mathbb{Q}} 0$ on the generic fibre of f. Then there exist resolutions $\phi\colon W \to X$ and $\psi\colon V \to Z$ such that the induced map $W \dashrightarrow V$ is a morphism and $\phi^* M \sim 0/V$.*

$$\begin{tikzcd} \phi^*M \subset W \arrow[r, dashed] \arrow[d, "\phi"'] & V \arrow[d, "\psi"] \\ M \subset X \arrow[r, "f"] & Z \end{tikzcd}$$

$M \subset X$, $M \sim_{\mathbb{Q}} 0$ on gen fibre of f

<u>Proof</u> (1) Show $\phi^* M \sim_{\mathbb{Q}} 0 \,/V$.

Since M is nef and $M \sim_{\mathbb{Q}} 0$ on gen fibre, $M \sim_{\mathbb{Q}} N$ for some N s.t components of N do not intersect the gen fibre.

Thus if A is the pullback of a sufficiently ample div on Z, then $\kappa(A+M) = \dim Z = \kappa(A-M)$.

{Numerical dimension: if D nef, define $\nu(D) := \max\{e \in \mathbb{Z}_{\geq 1} \mid D^e \not\equiv 0\}$.
We have $0 \leq \nu(D) \leq d$, $\kappa(D) \leq \nu(D)$. We say D is good if $\kappa(D) = \nu(D)$.

$\Rightarrow \dots \Rightarrow$ $A+M$ is nef and good. $\Rightarrow$ $\phi^* M \sim_{\mathbb{Q}} 0 \,/V$.

2.45. **Pairs with large boundaries.**

Lemma 2.46. *Let (X, B) be a projective $\mathbb{Q}$-factorial dlt pair of dimension d, and let M be a nef Cartier divisor. Let $a > 2d$ be a real number. Then any MMP on $K_X + B + aM$ is M-trivial, i.e. the extremal rays in the process intersect M trivially. If M is big, then $K_X + B + aM$ is also big.*

2.47. **Divisors with log discrepancy close to 0.**

Lemma 2.48. *Let $d \in \mathbb{N}$ and $\Phi \subset [0,1]$ be a DCC set. Then there is $\epsilon > 0$ depending only on d and Φ such that if (X, B) is a projective pair and D is a prime divisor on birational models of X satisfying*

- *(X, B) is lc of dimension d and $(X, 0)$ is klt,*
- *$K_X + B \sim_{\mathbb{R}} 0$ and $B \in \Phi$, and*
- *$a(D, X, B) < \epsilon$,*

then $a(D, X, B) = 0$.

<u>Proof</u> ACC for LCTs.

2.49. **Boundary coefficients close to 1.**

Proposition 2.50. *Let $d, p \in \mathbb{N}$ and let $\Phi \subset [0,1]$ be a DCC set. Then there is $\epsilon \in \mathbb{R}^{>0}$ depending only on d, p, Φ satisfying the following. Let $(X', B' + M')$ be a generalised pair of dimension d with data $\phi\colon X \to X' \to Z$ and M such that*

- *$B' \in \Phi \cup (1-\epsilon, 1]$ and pM is b-Cartier,*
- *$-(K_{X'} + B' + M')$ is a limit of movable/Z $\mathbb{R}$-divisors,*
- *there is*
$$0 \le P' \sim_{\mathbb{R}} -(K_{X'} + B' + M')/Z$$
such that $(X', B' + P' + M')$ is generalised lc, and
- *X' is $\mathbb{Q}$-factorial of Fano type/Z.*

Let Θ' be the boundary whose coefficients are the same as B' except that we replace each coefficient in $(1-\epsilon, 1)$ with 1. That is, $\Theta' = (B')^{\le 1-\epsilon} + \lceil (B')^{>1-\epsilon} \rceil$. Run an MMP/$Z$ on $-(K_{X'} + \Theta' + M')$ and let X'' be the resulting model. Then:

(1) *$(X', \Theta' + M')$ is generalised lc,*
(2) *the MMP does not contract any component of $\lfloor \Theta' \rfloor$,*
(3) *$-(K_{X''} + \Theta'' + M'')$ is nef/Z, and*
(4) *$(X'', \Theta'' + M'')$ is generalised lc.*